

The Instrumental Variable Method for Estimating Local Average Treatment Regime Effects

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Abstract

We propose the instrumental variable regime (IVR) method to estimate the causal effects of multiple sequential treatments. This method serves to address the problem of endogenous selections of sequential treatments. An IVR is a sequence of instrumental variables in which each IV instruments for an endogenous treatment variable. Our proposed method generalizes the LATE model in Imbens and Angrist (1994) from a single treatment to many treatments applied sequentially. More precisely, with the IVR this model allows for estimating the local average treatment regime effects (LATRE), possibly conditional on a set of initial covariates. Though there exist studies in this area that use IVR, all of them require a structural functional form assumption. Our method is novel in that we do not require any such assumption. Thus unlike previous approaches, ours is robust to model misspecifications, which usually occur in treatment regime settings. The ideas and estimators in this paper are motivated and illustrated through a contextual example showing the use of IVR

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to estimate the treatment regime effect of an honor program on student performance when this program is in use for multiple years. We use simulations to demonstrate the performance of the proposed method.

Keywords: instrumental variable regime, treatment regime, endogenous selection, no structural model, causal effects

1 Introduction

Estimating treatment effects has been a major concern among researchers across different fields including economics, statistics, epidemiology, sociology, etc. Within this research area, estimating treatment regime effects (i.e., effects of a set of sequential treatments over time) has drawn a lot of attention from scientists and practitioners. In this paper, we contribute to this large literature by developing an instrumental variable regime (IVR) framework to estimate the effect of a treatment sequence on a pre-defined outcome when the treatment selections are endogenous. The IVR is a set of instrumental variables each of which instruments for an endogenous treatment variable. Technically, we generalize the idea of the LATE model in Imbens and Angrist (1994) from a single treatment to including many sequential ones. Our proposed model also allows for the estimation of a form of local average treatment effect of a treatment regime. More similarly to Abadie (2003), we allow conditioning on a set of initial covariates of the treatment effect.

In experimental studies, we can control this endogeneity to stay away from it. However, experimental data are often difficult and expensive to obtain. Therefore researchers have shifted their focus to observational studies, unfortunately in which case the selection endogeneity occurs more often than not: individuals usually make decisions about treatments based on the evaluation of future potential outcomes. It is important to resolve this issue because otherwise, estimations of causal effects will be affected by a part generated by the selection process rather than only the true treatment effects. This mis-estimation invalidates all causal inference decisions.

There has been a lot of research on the use of IV in studying causal effects, some of which address the endogeneity problem in the sequential treatment (or longitudinal) setting (see, e.g., Hogan and Lancaster (2004); O'Malley (2012); Sitlani et al. (2012); Wooldridge

(2002)). However, to the best of our knowledge all of these precedent methods require a structural functional form assumption on the potential outcomes. In the treatment regime setting, this assumption can hardly be valid and thus, these methods can easily suffer from model misspecifications. In contrast, our proposed method does not make any such assumption and is therefore free from the model misspecification issue.

We present our ideas in the context of an evaluation of the effect of an Honors Program on students' performances when this program is in use for multiple consecutive years. In year $T = 0$, students observe their initial information X_0 . Then they make decisions about whether to follow the honors program ($W_0 = 1$) or the standard program ($W_0 = 0$). In year $T = 1$, students observe the outcome Y_1 and other information X_1 . Then they make decisions about whether to switch or to stay in the current program ($W_1 \in \{0, 1\}$). In year $T = 2$, they observe new outcome Y_2 and other information X_2 . We can think of X_i as a set of personal information such as gender, parents' income, and age. The outcome Y_i can be Math test score.

The endogenous treatment selection occurs as students may anticipate the future outcome of each program, thus biasing their decisions about which program to join in each year. Hence in this setting the sequential randomization assumption made in traditional treatment regime literature (see e.g., Orellana et al. (2010); Hogan and Lancaster (2004); Pham (2016); Wooldridge (2002)) fails to be true; the identification results are therefore no longer valid.

Our proposed solution is to introduce a set of instrumental variables or an IVR Z_i 's where each Z_i instruments for W_i . In this setting, Z_i could be the distances from students' houses to school after discretizing the values into 1 or 0 corresponding to far or close. Another choice of Z_i could be the indicator that the students subscribe to a Physics magazine. In other settings, the choice of Z_i 's varies depending on the knowledge of the experts in

the field.

As mentioned earlier, there is a literature on how to use IV's to estimate treatment regime effects (see, e.g., Hogan and Lancaster (2004); Wooldridge (2002)). However, all of them require a (usually linear) functional form assumption on the potential outcomes, and using IV's in the same way as we deal with linear regression models. Unlike them, we use IVs combined with propensity scores in a way that does not need to make such a functional form assumption; thus, our method is robust to (outcome) model misspecifications.

With the introduction of the IVR Z_i 's, we can estimate the local average treatment regime effects (LATRE), possibly conditional on a set of initial covariates. This set of initial covariates is significant in terms of usage. For example in our setting above, we can condition on the gender variable. Then, we can estimate the LATRE for female students and for male students separately.

The rest of the paper is organized as follows. Section 2 describes the instrumental variable regime model. Section 3 discusses the identification results for the two-period model. Section 4 gives main identification results about the estimation of the local average treatment regime effect, perhaps conditional on a set of initial covariates. Section 5 focuses on the application of the model. Specifically, we run the proposed framework on a two-period setting using a simulated data set. Section 6 reviews related literature and concludes.

2 The Instrumental Variable Regime Model

The Instrumental Variable Regime (IVR) model is built upon the Treatment Regime model (see e.g., Chakraborty and Moodie (2013); Murphy et al. (2001); Orellana et al. (2010); Pham (2016)) and the LATE model (see Imbens and Angrist (1994); Abadie (2003); Angrist et al. (1996)). This model addresses the endogeneity problem in the selections of

treatments when the treatments are applied sequentially in multiple periods. The model is specified as follows.

There are N students and $(T + 1)$ periods where $T \geq 1$.¹ Each student i 's data is comprised of observations

$$(X_0^i, W_0^i, Y_1^i, X_1^i, W_1^i, \dots, Y_T^i, X_T^i, W_T^i, Y_{T+1}^i, X_{T+1}^i).$$

Here, X_0^i is the initial set of covariates; for $j \in \{1, \dots, T + 1\}$, X_j^i is the set of covariates in period j after receiving treatment W_{j-1}^i but before receiving treatment W_j^i . For $j \in \{0, 1, \dots, T\}$, W_j^i is the treatment whose value, that is, treatment level in period j belongs to the set \mathcal{W}_j^i . In this setting, we assume $\mathcal{W}_j^i = \{0, 1\}$ for each j . After making a treatment decision in period $j \in \{0, 1, \dots, T\}$, there is an observed outcome in period $(j + 1)$ denoted by Y_{j+1}^i . We assume that X_j^i 's are exogenous and Y_j^i 's are functions of only student i 's past information. That is, we assume away the interference effects. These effects exist in many settings, but we do not address them in this paper. In the latter parts of this paper, we will suppress the index i when there is no ambiguity.

We proceed by defining

$$W_j = 0 \text{ for } j < 0; X_j = 0 \text{ for } j < 0; Y_j = 0 \text{ for } j < 1.$$

Also,

$$O_0 = X_0; \text{ and } O_j = (Y_j, X_j) \text{ for all } j \in \{1, 2, \dots, T + 1\}.$$

We also use overbars with a subscript j to denote the present variable at time j and all its past values. For example, $\overline{O}_j = (O_0, O_1, \dots, O_j)$. We use notations with no subscript to denote the whole history. For example, $O = \overline{O}_{T+1}$. Moreover, as discussed above Y_{j+1} is a

¹When $T = 0$, we have the single treatment case with variables (X_0, W_0, Y_1) .

function of \overline{O}_j and \overline{W}_j ; W_j is a function of \overline{O}_j ; we suppress the dependency on \overline{O}_j in the latter discussion.

Among many, Orellana et al. (2010); Pham (2016) make the sequential randomization assumption that each W_j is independent of all potential outcomes given past information $(\overline{O}_j, \overline{W}_{j-1})$ and derives heterogeneous treatment regime effects. Moreover by taking the expectation, he could easily obtain average treatment regime effects. However in many settings including ours, this assumption is violated. To be more precise, each student's treatment decision is often driven by the expected potential outcome; students might believe honors program will be beneficial so their choices will be biased toward it.

In this paper, we explicitly consider this case in which the selections of treatments are endogenous. Hence, the sequential randomization assumption in the previous literature no longer holds here.

This relaxation would make all the identification results (e.g., in Orellana et al. (2010); Pham (2016)) fail to be true. To address this issue, we propose the use of an instrumental variable regime in a similar sense to the instrumental variable in the LATE model. Specifically, assume that there are $(T + 1)$ instrumental variables Z_j 's for $j = 0, 1, \dots, T$, one for each W_j . Let \mathcal{Z}_j be the domain for Z_j for each j . We assume further that each $\mathcal{Z}_j = \{0, 1\}$. In our motivational setting, Z_j could be students' distances to school when these values are discretized to 1 or 0 corresponding to being far or being close. Another choice could be whether students subscribe to a Physics magazine. This will not affect the potential outcome directly, but will certainly affect the decision to join the honors program or not.

Now for each possible realization $z = (z_0, z_1, \dots, z_T)$ of $Z = \overline{Z}_T$ and $w = (w_0, w_1, \dots, w_T)$ of $W = \overline{W}_T$, we define the vectors of potential outcomes:

$$W(z) = (W_0(z_0), W_1(z_1), \dots, W_T(z_T)) \text{ and}$$

$$O(z, w) = (X_0, Y_1(z_0, w_0), X_1(z_0, w_0), \dots, Y_{T+1}(\overline{z}_T, \overline{w}_T), X_{T+1}(\overline{z}_T, \overline{w}_T)).$$

We write

$$\mathcal{O} = \{(W(z), O(z, w)) \mid z_j \in \mathcal{Z}_j \text{ and } w_j \in \mathcal{W}_j \text{ for } j = 0, 1, \dots, T\}$$

to denote the set of all possible vectors of potential outcomes. More concretely, we make the following assumptions.

First, we implicitly assume that there is no interference between students: each student has his potential outcomes and his choice as a function of his past outcomes and instruments. Furthermore, a standard assumption is that the observed treatments and outcomes are consistent with the relevant potential treatments and outcomes.

Assumption 1. (*IVR Consistency*)

- (i) For each $j \in \{0, 1, \dots, T\}$, we have $W_j = W_j(Z_j)$.
- (ii) For each $j \in \{1, \dots, T, T+1\}$, we have $(Y_j, X_j) = (Y_j(\bar{Z}_{j-1}, \bar{W}_{j-1}), X_j(\bar{Z}_{j-1}, \bar{W}_{j-1}))$.

Next, we introduce a modified version of the sequential randomization assumption. We impose exclusion assumptions on Z_j 's to make them valid instruments.

Assumption 2. (*IVR Sequential Randomization*)

- (i) (*Independence of IVR*) For each $j \in \{0, 1, \dots, T\}$, Z_j is conditionally independent of \mathcal{O} given \bar{O}_j and \bar{Z}_{j-1} .
- (ii) (*Exclusion of IVR*)

– For each $j \in \{1, \dots, T, T+1\}$ and arbitrary $\bar{Z}_{j-1}, \bar{Z}'_{j-1}$, and \bar{W}_{j-1} , we have

$$(Y_j(\bar{Z}_{j-1}, \bar{W}_{j-1}), X_j(\bar{Z}_{j-1}, \bar{W}_{j-1})) = (Y_j(\bar{Z}'_{j-1}, \bar{W}_{j-1}), X_j(\bar{Z}'_{j-1}, \bar{W}_{j-1})).$$

– For each $j \in \{0, 1, \dots, T\}$,

$$\{(Y_j(\bar{z}_{j-1}, \bar{w}_{j-1}), W_j(z_j)) | z_i, w_i \in \{0, 1\} \text{ for } i = 0, \dots, j-1\} \perp Z_j.$$

Thanks to Assumption 2 (ii), we now can write $Y_j(\bar{W}_{j-1})$ instead of $Y_j(\bar{Z}_{j-1}, \bar{W}_{j-1})$ and similarly for X_j . Also, $O(z, w) = O(w)$ for each z and w .

Before proceeding, we make another assumption that each Z_j takes each value in its domain with positive probability. This assumption is essential as it necessitates the estimation of the causal effect of each treatment regime.

Assumption 3. (IVR Positivity) For each $j \in \{0, 1, \dots, T\}$ and each realization of \bar{O}_j and \bar{Z}_j , the following condition holds with probability 1:

$$0 < P(Z_j = 1 | \bar{O}_j, \bar{Z}_{j-1}) < 1.$$

Recall that our main interest is to estimate the average causal effect of a treatment regime (W_0, W_1, \dots, W_T) on some measurable function $u(\cdot)$ of the outcomes O . Let d denote a treatment regime that assigns each W_j to a fixed value in \mathcal{W}_j . We write $W^d = (W_0^d, W_1^d, \dots, W_T^d)$ to denote the treatment sequence if the subject had followed the regime d . Likewise, we write $O^d = O(W^d)$ to denote the vector of outcomes if the subject had followed the regime d . Then, the object of interest is $u(O^d)$. In many cases, $u(O^d) = Y_{T+1}$ or $u(O^d) = \sum_{t=1}^{T+1} Y_t$.

Now, we note that a realization \bar{Z}_T of an IVR is a vector in $\{0, 1\}^{T+1}$. For two realizations \bar{Z}_T and \bar{Z}'_T , we write

- $\bar{W}_T(\bar{Z}_T) \succeq \bar{W}_T(\bar{Z}'_T)$ if $W_j(Z_j) \geq W_j(Z'_j)$ for all j and there exists some k such that $W_k(Z_k) > W_k(Z'_k)$;
- $\bar{W}_T(\bar{Z}_T) \succ \bar{W}_T(\bar{Z}'_T)$ if $W_j(Z_j) > W_j(Z'_j)$ for all j .

Similarly to the Local Average Treatment Effect (LATE) model introduced by Imbens and Angrist (1994), we are able to obtain identification results in this dynamic setting.

With a single binary treatment variable W and a corresponding binary instrument Z , traditional assumption is the monotonicity assumption, which states $P(W(1) \geq W(0)|X) = 1$. We make a similar assumption here.

Assumption 4. (*Monotonicity*) For each $j \in \{0, 1, \dots, T\}$, we have

$$P(W_j(1) \geq W_j(0)) = 1.$$

The last independence assumption we need is the independence between W_j 's in different periods conditional on common past information.

Assumption 5. For any $j, k \in \{0, 1, \dots, T\}$ with $j \neq k$, we have

$$(W_j(0), W_j(1)) \perp (W_k(0), W_k(1)) \mid \overline{O}_{\min(j,k)}, \overline{Z}_{\min(j,k)-1}.$$

In the next section, we discuss the identification results for the two-period case.

3 Identification Results For Two-Period Model

We consider the case with $T = 1$. Denote $u(\cdot) = u(X_0, W_0, Y_1, X_1, W_1, Y_2)$. Denote $u_{ij}(\cdot) = u(X_0, i, Y_1, X_1, j, Y_2)$. We are interested in the identification result of $\mathbb{E}[u(\cdot)|X_0, W_j(1) > W_j(0) \forall j = 0, 1]$ and $\mathbb{E}[u_{ij}(\cdot)|X_0, W_j(1) > W_j(0) \forall j = 0, 1]$.

3.1 Local Identification Results

By DeMorgan's Law, we have

$$\begin{aligned}
& P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0) \times \mathbb{E}[u(\cdot) | X_0, W_j(1) > W_j(0) \forall j = 0, 1] \\
&= \mathbb{E}[u(\cdot) | X_0] \\
&\quad - \sum_{i=0,1} \mathbb{E}[u(\cdot) | X_0, W_0(1) = W_0(0) = i] P(W_0(1) = W_0(0) = i | X_0) \tag{1}
\end{aligned}$$

$$\begin{aligned}
&\quad - \sum_{i=0,1} \mathbb{E}[u(\cdot) | X_0, W_0(1) > W_0(0), W_1(1) = W_1(0) = i] \\
&\quad \times P(W_0(1) > W_0(0) | X_0) P(W_1(1) = W_1(0) = i | X_0). \tag{2}
\end{aligned}$$

Thus, in order to identify $\mathbb{E}[u(\cdot) | X_0, W_j(1) > W_j(0) \forall j = 0, 1]$ we need to identify three terms: Sum (1), Sum (2), and $P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0)$; see appendix for derivation. Combining these identification results, we obtain

$$\mathbb{E}[u(\cdot) | X_0, W_j(1) > W_j(0)] = \frac{\mathbb{E}[\kappa u(\cdot) | X_0]}{P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0)}, \tag{3}$$

where

$$\begin{aligned}
\kappa = & 1 - \frac{W_0(1 - Z_0)}{P(Z_0 = 0 | X_0, X_1)} - \frac{(1 - W_0)Z_0}{P(Z_0 = 1 | X_0, X_1)} - \frac{W_1(1 - Z_1)}{P(Z_1 = 0 | X_0, X_1)} - \frac{(1 - W_1)Z_1}{P(Z_1 = 1 | X_0, X_1)} \\
& + \frac{W_0(1 - Z_0)W_1(1 - Z_1)}{P(Z_1 = 0, Z_0 = 0 | X_0, X_1)} + \frac{(1 - W_0)Z_0W_1(1 - Z_1)}{P(Z_1 = 0, Z_0 = 1 | X_0, X_1)} \\
& + \frac{W_0(1 - Z_0)(1 - W_1)Z_1}{P(Z_1 = 1, Z_0 = 0 | X_0, X_1)} + \frac{(1 - W_0)Z_0(1 - W_1)Z_1}{P(Z_1 = 1, Z_0 = 1 | X_0, X_1)}
\end{aligned}$$

3.2 Local Average Treatment Regime Effect

We can then express local average treatment regime effect in terms of Equation (3). For example, the local average effect of a regime switching W_0 from 0 to 1 and W_1 from 1 to

0, for a complier, for whom $W_j = Z_j$, is

$$\begin{aligned} & \mathbb{E}\left[u_{10}(\cdot) - u_{01}(\cdot) | X_0, W_j(1) > W_j(0) \forall j = 0, 1\right] \\ = & \frac{1}{P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0)} \left[\mathbb{E}\left[\kappa u(\cdot) \frac{W_0(1 - W_1)}{P(Z_0 = 1, Z_1 = 0 | X_0, X_1)} | X_0\right] \right. \\ & \left. - \mathbb{E}\left[\kappa u(\cdot) \frac{(1 - W_0)W_1}{P(Z_0 = 0, Z_1 = 1 | X_0, X_1)} | X_0\right] \right]. \quad (4) \end{aligned}$$

3.3 Unconditional Local Identification

As X_0 might typically has high dimensions, it is convenient to illustrate the local average treatment regime effect unconditionally. Now let's derive the expression for unconditional effect which we will use in simulation part. Applying Bayes' theorem and integrating yields

$$\begin{aligned} & \int \mathbb{E}\left[u(\cdot) | X_0, W_j(1) > W_j(0)\right] dP(X_0 | W_j(1) > W_j(0) \forall j = 0, 1) \\ = & \int \mathbb{E}[\kappa u(\cdot) | X_0] \frac{dP(X_0 | W_j(1) > W_j(0) \forall j = 0, 1)}{P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0)} \\ = & \int \mathbb{E}[\kappa u(\cdot) | X_0] \frac{P(X_0, W_j(1) > W_j(0) \forall j = 0, 1) dX_0 / P(W_j(1) > W_j(0) \forall j = 0, 1)}{P(X_0, W_j(1) > W_j(0) \forall j = 0, 1) / P(X_0)} \\ = & \frac{1}{P(W_j(1) > W_j(0) \forall j = 0, 1)} \int \mathbb{E}[\kappa u(\cdot) | X_0] dP(X_0). \quad (5) \end{aligned}$$

Plugging Equation (3.4) into Equation (3.5), we obtain the unconditional effect for a complier with regime switch W_0 from 0 to 1 and W_1 from 1 to 0

$$\begin{aligned} & \mathbb{E}\left[u_{10}(\cdot) - u_{01}(\cdot) | W_j(1) > W_j(0) \forall j = 0, 1\right] \\ = & \frac{\mathbb{E}[g(X_0)]}{P(W_j(1) > W_j(0) \forall j = 0, 1)}, \quad (6) \end{aligned}$$

where

$$g(X_0) = \mathbb{E}\left[\kappa u(\cdot) \frac{W_0(1 - W_1)}{P(Z_0 = 1, Z_1 = 0 | X_0, X_1)} | X_0\right] - \mathbb{E}\left[\kappa u(\cdot) \frac{(1 - W_0)W_1}{P(Z_0 = 0, Z_1 = 1 | X_0, X_1)} | X_0\right].$$

Next, we move on to the identification results for the general case. For the more general case, we only present the results for conditional effect; results for unconditional effect can be derived from the conditional version analogously.

4 Main Identification Results

We denote $u(\cdot) = u(X_0, W_0, Y_1, X_1, W_1, \dots, X_T, W_T, Y_{T+1})$. Let's start with the identification for local treatment effect on a full complier, for whom $W_j(1) > W_j(0)$ for all periods j . DeMorgan's Law gives

$$\begin{aligned}
& P\left(W_j(1) > W_j(0) \forall j | X_0\right) \times \mathbb{E}\left[u(\cdot) | X_0, W_j(1) > W_j(0) \forall j\right] \\
= & \mathbb{E}\left[u(\cdot) | X_0\right] \\
& - \sum_{i \in \{0,1\}} \mathbb{E}\left[u(\cdot) | X_0, W_0(1) = W_0(0) = i\right] \times P\left(W_0(1) = W_0(0) = i | X_0\right) \\
& - \sum_{\tau=0}^{T-1} \sum_{i \in \{0,1\}} \mathbb{E}\left[u(\cdot) | X_0, W_j(1) > W_j(0), \forall j \leq \tau, W_{\tau+1}(1) = W_{\tau+1}(0) = i\right] \\
& \quad \times \left(\prod_{j=0}^{\tau} P(W_j(1) > W_j(0) | X_0)\right) \times P\left(W_{\tau+1}(1) = W_{\tau+1}(0) = i | X_0\right).
\end{aligned}$$

Therefore, in order to identify $\mathbb{E}\left[u(\cdot) | X_0, W_j(1) > W_j(0) \forall j\right]$ we need to identify the sums on the right hand side of the above equality and the conditional probabilities $P\left(W_j(1) > W_j(0) | X_0\right)$.

4.1 First Step Identification

We denote $K_{t,0} = W_t(1 - Z_t)$ and $K_{t,1} = (1 - W_t)Z_t$. We postpone the identification of conditional probabilities for a moment. We apply DeMorgan's Law to evaluate the expected

utility of a full-complier:

$$\begin{aligned} & \mathbb{E}[u(\cdot)|X_0, W_j(1) > W_j(0) \forall j] \\ &= \frac{\mathbb{E}[\kappa u(\cdot)|X_0]}{P(W_j(1) > W_j(0) \forall j|X_0)}, \end{aligned} \quad (7)$$

where

$$\kappa = 1 + \sum_{\tau=1}^T (-1)^\tau \sum_{\substack{i_1, \dots, i_\tau \\ \in \{0,1\}}} \sum_{\substack{j_1 < \dots < j_\tau \\ \in \{0,1, \dots, T\}}} \frac{\prod_{t=1}^{\tau} K_{j_t, i_t}}{P(Z_{j_1} = i_1, \dots, Z_{j_\tau} = i_\tau | X_0, \dots, X_{j_\tau})}.$$

In one treatment case, we cannot learn anything about non-compliers; in contrast, we can learn about local average treatment effects for people of different compliance types. Indeed, program researchers may be more interested in treatment effects on different compliance types other than the full-compliance. To this end, denote the periods of compliance by

$$\mathcal{T}_c := \{j \in \{0, \dots, T\} | W_j(1) > W_j(0)\},$$

and the periods of non-compliance by

$$\mathcal{T}_n^0 := \{j \in \{0, \dots, T\} | W_j(1) = W_j(0) = 0\}, \text{ and}$$

$$\mathcal{T}_n^1 := \{j \in \{0, \dots, T\} | W_j(1) = W_j(0) = 1\}.$$

Note that under the monotonicity assumption 4, we can consider only the compliance type. Then a compliance type can be represented by a tuple $(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1)$. For instance, the full-compliance type is represented by the tuple $(\{0, \dots, T\}, \emptyset, \emptyset)$. We define a factor κ associated with compliance type $(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1)$ by

$$\begin{aligned} \kappa &= \frac{\prod_{t \in \mathcal{T}_n^0} K_{t,0} \prod_{t \in \mathcal{T}_n^1} K_{t,1}}{P(Z_j = i \forall j \in \mathcal{T}_n^i, i = 0, 1 | X_0)} \times \\ &\quad \left(1 + \sum_{\tau=1}^{|\mathcal{T}_c|} (-1)^\tau \sum_{\substack{i_1, \dots, i_\tau \\ \in \{0,1\}}} \sum_{\substack{j_1 < \dots < j_\tau \\ \in \mathcal{T}_c}} \frac{\prod_{t=1}^{\tau} K_{j_t, i_t}}{P(Z_{j_1} = i_1, \dots, Z_{j_\tau} = i_\tau | X_0, \dots, X_{j_\tau})} \right), \end{aligned}$$

where $\kappa = \kappa(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1, X_0)$ is a function of compliance type and period-0 covariates. When there is no ambiguity, we leave out the type and covariate arguments for κ . As a general result, the expected utility of an agent with compliance type $(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1)$ is

$$\mathbb{E} [u(\cdot)|X_0, (\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1)] = \frac{\mathbb{E} [\kappa u(\cdot)|X_0]}{P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0)}. \quad (8)$$

The identification results will be complete once we can identify

$$\begin{aligned} & P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0) \\ \equiv & P(W_j(1) > W_j(0) \forall j \in \mathcal{T}_c \text{ \& } W_j(1) = W_j(0) = 0 \forall j \in \mathcal{T}_n^0 \text{ \& } W_j(1) = W_j(0) = 1 \forall j \in \mathcal{T}_n^1|X_0), \end{aligned}$$

which is the focus of the next section.

4.2 Main Probability Estimation

Let us first consider the full-compliance case, that is, $\mathcal{T}_c = \{0, \dots, T\}$, keeping in mind that the general case is similar. In other words, we consider

$$P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0) = P(W_j(1) > W_j(0) \forall j|X_0).$$

Now under Assumption 5, we have (see the iterative derivation in Appendix)

$$P(W_j(1) > W_j(0) \forall j|X_0) = \prod_{j=0}^T P(W_j(1) > W_j(0)|X_0). \quad (9)$$

Similarly, for any set $(i_0, \dots, i_T) \in \{0, 1\}^{T+1}$ we have

$$\mathbb{E} \left[\prod_{j=0}^T W_j(i_j) \middle| X_0 \right] = P(W_j(i_j) = 1 \forall j|X_0) = \prod_{j=0}^T P(W_j(i_j) = 1|X_0) = \prod_{j=0}^T \mathbb{E}[W_j(i_j)|X_0].$$

On the other hand by Assumption 4,

$$\begin{aligned}
P(W_j(1) > W_j(0)|X_0) &= 1 - P(W_j(0) = 1|X_0) - P(W_j(1) = 0|X_0) \\
&= P(W_j(1) = 1|X_0) - P(W_j(0) = 1|X_0) \\
&= \mathbb{E}[W_j(1)|X_0] - \mathbb{E}[W_j(0)|X_0].
\end{aligned}$$

Therefore, we can rewrite $P(W_j(1) > W_j(0) \forall j|X_0)$ as

$$\prod_{j=0}^T \left(\mathbb{E}[W_j(1)|X_0] - \mathbb{E}[W_j(0)|X_0] \right) = \sum_{(i_0, \dots, i_T) \in \{0,1\}^{T+1}} (-1)^{T+1-\sum_{j=0}^T i_j} \cdot \mathbb{E} \left[\prod_{j=0}^T W_j(i_j) \middle| X_0 \right].$$

Thus, we can determine the probability $P(W_j(1) = 1, W_j(0) = 0 \forall j|X_0)$ if each term $\mathbb{E} \left[\prod_{j=0}^T W_j(i_j) \middle| X_0 \right]$ is determined. This can be done according to the following lemma.

Lemma 1. *Fix $(i_0, \dots, i_T) \in \{0,1\}^{T+1}$. Then under Assumptions 1, 2, 3, we have*

$$\mathbb{E} \left[\prod_{j=0}^T W_j \cdot \prod_{j=0}^T \frac{\mathbf{1}_{\{Z_j=i_j\}}}{P(Z_j=i_j|\overline{O}_j, \overline{Z}_{j-1})} \middle| X_0 \right] = \mathbb{E} \left[\prod_{j=0}^T W_j(i_j) \middle| X_0 \right].$$

Proof. Lemma 1 is indeed a special case of Theorem 3.1 in Pham (2016). \square

Lemma 1 and the above reasoning implies Theorem 1 below, which is the identification result for $P(W_j(1) > W_j(0) \forall j|X_0)$.

Theorem 1. *Under Assumptions 1, 2, 3, and 5, we have*

$$P(W_j(1) > W_j(0) \forall j|X_0) = \mathbb{E} \left[\prod_{j=0}^T W_j \cdot \sum_{(i_0, \dots, i_T) \in \{0,1\}^{T+1}} \prod_{j=0}^T \frac{(-1)^{1-i_j} \cdot \mathbf{1}_{\{Z_j=i_j\}}}{P(Z_j=i_j|\overline{O}_j, \overline{Z}_{j-1})} \middle| X_0 \right].$$

In a similar manner, we can fully identify $P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0)$ in the general case. See proof in the appendix.

Theorem 2. *Under Assumptions 1, 2, 3, and 5, we have*

$$P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1 | X_0) = \mathbb{E} \left[\prod_{j \in \mathcal{T}_c \cup \mathcal{T}_n^1} W_j \cdot \prod_{j \in \mathcal{T}_n^0} (1 - W_j) \cdot \left(\sum_{(i_j)_{j \in \mathcal{T}_c} \in \{0,1\}^{|\mathcal{T}_c|}} \prod_{j \in \mathcal{T}_c} \frac{(-1)^{1-i_j} \cdot \mathbf{1}_{\{Z_j=i_j\}}}{P(Z_j = i_j | \bar{O}_j, \bar{Z}_{j-1})} \right) \right. \\ \left. \cdot \prod_{j \in \mathcal{T}_n^0} \frac{\mathbf{1}_{\{Z_j=1\}}}{P(Z_j = 1 | \bar{O}_j, \bar{Z}_{j-1})} \prod_{j \in \mathcal{T}_n^1} \frac{\mathbf{1}_{\{Z_j=0\}}}{P(Z_j = 0 | \bar{O}_j, \bar{Z}_{j-1})} \middle| X_0 \right].$$

5 Application

To identify or apply our above results to estimate local treatment effects, one needs to know the compliance type of each observation in the sample. Here are a few scenarios that are ideal for application: if the researcher has good institutional knowledge that helps him classify the compliance type based on each subject's covariates; if the whole sample consists of a single compliance type so that one can easily identify the compliance type based on the realized treatment regimes. The following simulation exercise shows how to apply our results to a full compliance environment.

5.1 Simulation Setup

For this section, we consider a two-period setting with $T = 1$. The outcome of interest is $u(\cdot) = Y_{T+1} \equiv Y_2$.

The simulation procedure consists of repeating the following process 500 times.

We generate $n = 500,000$ observations of the following variables.

1. Simulate $X_0 \in U[-1, 1]^6$.
2. Set $X_1 \equiv X_0$.
3. Simulate $\epsilon_0 \sim U[0, 1]$ and $Z_0 \in \{0, 1\}$ with $P(Z_0 = 1 | X_0) = 1/(1 + \exp(X_0 \xi))$.

4. Simulate $W_0 \in \{0, 1\}$ such that

$$P(W_0 = 1) = \begin{cases} \epsilon_0 & \text{if } Z_0 = 0 \\ 1 & \text{if } Z_0 = 1 \end{cases}.$$

5. Generate

$$Y_1 = X_0\alpha_1 + \beta_1 W_0 + \epsilon_0.$$

6. Simulate $\epsilon_1 \sim U[0, 1]$ and $Z_1 \in \{0, 1\}$ with $P(Z_1 = 1) = e_1 \in (0, 1)$.

7. Simulate $W_1 \in \{0, 1\}$ such that

$$P(W_1 = 1) = \begin{cases} \epsilon_1 & \text{if } Z_1 = 0 \\ 1 & \text{if } Z_1 = 1 \end{cases}.$$

8. Generate

$$Y_2 = X_0\alpha_2 + \beta_2 W_0 + \delta Y_1 + \gamma W_1 + \epsilon_1.$$

So

$$Y_2 = X_0(\alpha_2 + \delta\alpha_1) + (\beta_2 + \delta\beta_1)W_0 + \gamma W_1 + \delta\epsilon_0 + \epsilon_1$$

Here, the parameters are chosen to be

- $\xi = [1, 2, 3, -1, -2, -3]^T$; $e_1 = 0.75$
- $\alpha_1 = [1, 1, 1, 1, 1, 2, 2, 2, 2, 2]^T$; $\beta_1 = 2$.
- $\alpha_2 = [2, 2, 2, 2, 2, 1, 1, 1, 1, 1]^T$; $\beta_2 = 2$; $\delta = 2$; $\gamma = 1$.

5.2 Simulation Result

We want to estimate the local effect of regime (1, 0) with respect to regime (0, 1). Under our assumptions, the true conditional (local) average treatment regime effect is $\tau = (\beta_2 + \delta\beta_1) - \gamma = 5$.

We compare our proposed method with two other methods:

- Naive approach: We take the difference in means of two sets of observations corresponding to regime (1, 0) and regime (0, 1).
- No IVs: We assume sequential treatments, but do not use IVs. The formula will be similar to Equation (6) except that in the numerator, we use the formula in Pham (2016) with only W_0, W_1 instead of Z_0, Z_1 .

We compare these methods using the 500 simulated datasets generated above in terms of four measures:

- absolute mean error: $|\mathbb{E}[\hat{\tau} - \tau]|$.
- mean absolute error: $\mathbb{E}[|\hat{\tau} - \tau|]$.
- absolute median error: $|median(\hat{\tau} - \tau)|$.
- median absolute error: $median(|\hat{\tau} - \tau|)$.

Here, τ is the true outcome ($\tau = 5$) while $\hat{\tau}$ is an estimator of τ . The results are summarized in Table 1.

As we can see, if we do not use IVR when the problem of endogenous selections of sequential treatments is present then the estimate is terrible. In this simulation study, it is

Error Metric	Absolute	Mean	Absolute	Median
	Mean	Absolute	Median	Absolute
Naive	0.86	0.86	0.86	0.86
No IV	1.24	1.24	1.24	1.24
LATRE	0.56	0.56	0.54	0.54

Table 1: Simulation Results. (Smaller is better.)

even worse than the naive estimate. The use of IVR (that is, our method LATRE) makes the estimation significantly better, which outperforms the naive estimate.

6 Conclusions

Following Murphy et al. (2001); Orellana et al. (2010); Pham (2016) among others, we attack the setting with a treatment sequence. Generalizing the LATE model of Imbens and Angrist (1994), we provide a method of using IVR to estimate the local average treatment regime effects. Though there are many research studies regarding the LATE model (notably Imbens and Angrist (1994); Angrist et al. (1996); Abadie (2003)), we are not aware of any work regarding the LATRE model before ours.

Our method is shown to work well in the simulation above in a two-treatment setting, but it would also be useful with a longer sequence of treatments. Future work would be applying our proposed method to experimental and non-experimental datasets.

Appendix A Derivation of Equation (3.3)

A.1 First Sum Identification

By Bayes rule we have

$$\begin{aligned}
& \mathbb{E} [W_0(1 - Z_0)u(\cdot)|X_0] \\
= & \mathbb{E} [u(\cdot)|X_0, W_0 = 1, Z_0 = 0] P(W_0 = 1|X_0, Z_0 = 0)P(Z_0 = 0|X_0); \\
& \mathbb{E} [(1 - W_0)Z_0u(\cdot)|X_0] \\
= & \mathbb{E} [u(\cdot)|X_0, W_0 = 0, Z_0 = 1] P(W_0 = 0|X_0, Z_0 = 1)P(Z_0 = 1|X_0).
\end{aligned}$$

By independence assumption and the above lemma, we can rewrite the summation (1) as

$$\frac{\mathbb{E} [W_0(1 - Z_0)u(\cdot)|X_0]}{P(Z_0 = 0|X_0)} + \frac{\mathbb{E} [(1 - W_0)Z_0u(\cdot)|X_0]}{P(Z_0 = 1|X_0)}.$$

A.2 Second Sum Identification

Similarly we have

$$\begin{aligned}
& \mathbb{E} [u(\cdot)|X_0, X_1, W_1(1) = W_1(0) = 1] P(W_1(1) = W_1(0) = 1|X_0, X_1) \\
= & \frac{\mathbb{E} [W_1(1 - Z_1)u(\cdot)|X_0, X_1]}{P(Z_1 = 0|X_0, X_1)}; \\
& \mathbb{E} [u(\cdot)|X_0, X_1, W_1(1) = W_1(0) = 0] P(W_1(1) = W_1(0) = 0|X_0, X_1) \\
= & \frac{\mathbb{E} [(1 - W_1)Z_1u(\cdot)|X_0, X_1]}{P(Z_1 = 1|X_0, X_1)}.
\end{aligned}$$

Note that we can express conditional analogs of terms in summation (2) as

$$\begin{aligned}
& \mathbb{E}[u(\cdot)|X_0, X_1, W_0(1) > W_0(0), W_1(1) = W_1(0) = i] \times P(W_0(1) > W_0(0)|X_0, X_1) \\
= & \mathbb{E}[u(\cdot)|X_0, X_1, W_1(1) = W_1(0) = i] \\
& - \sum_j \mathbb{E}[u(\cdot)|X_0, X_1, W_0(1) = W_0(0) = j, W_1(1) = W_1(0) = i] \\
& \times P(W_0(1) = W_0(0) = j|X_0, X_1).
\end{aligned}$$

By Bayes rule we have

$$\begin{aligned}
& \mathbb{E}[W_0(1 - Z_0)W_1(1 - Z_1)u(\cdot)|X_0, X_1] \\
= & \mathbb{E}[u(\cdot)|X_0, X_1, W_0 = W_1 = 1, Z_0 = Z_1 = 0] \\
& \times P(W_0 = W_1 = 1|X_0, X_1, Z_0 = Z_1 = 0)P(Z_0 = Z_1 = 0|X_0, X_1).
\end{aligned}$$

Then using the independence assumption, we can rewrite the summation (2) as

$$\begin{aligned}
& \mathbb{E}_{X_1|X_0} \left[\frac{\mathbb{E}[W_1(1 - Z_1)u(\cdot)|X_0, X_1]}{P(Z_1 = 0|X_0, X_1)} + \frac{\mathbb{E}[(1 - W_1)Z_1u(\cdot)|X_0, X_1]}{P(Z_1 = 1|X_0, X_1)} \right. \\
& - \frac{\mathbb{E}[W_0(1 - Z_0)W_1(1 - Z_1)u(\cdot)|X_0, X_1]}{P(Z_1 = 0, Z_0 = 0|X_0, X_1)} - \frac{\mathbb{E}[(1 - W_0)Z_0W_1(1 - Z_1)u(\cdot)|X_0, X_1]}{P(Z_1 = 0, Z_0 = 1|X_0, X_1)} \\
& \left. - \frac{\mathbb{E}[W_0(1 - Z_0)(1 - W_1)Z_1u(\cdot)|X_0, X_1]}{P(Z_1 = 1, Z_0 = 0|X_0, X_1)} - \frac{\mathbb{E}[(1 - W_0)Z_0(1 - W_1)Z_1u(\cdot)|X_0, X_1]}{P(Z_1 = 1, Z_0 = 1|X_0, X_1)} \right].
\end{aligned}$$

A.3 Probability Identification

Under Assumption 5, we have

$$\begin{aligned}
& P(W_j(1) > W_j(0) \forall j = 0, 1|X_0) \\
= & P(W_0(1) > W_0(0)|X_0) \times P(W_1(1) > W_1(0)|W_0(1) > W_0(0), X_0) \\
= & P(W_0(1) > W_0(0)|X_0) \times P(W_1(1) > W_1(0)|X_0).
\end{aligned}$$

Similarly, for any set $(i_0, i_1) \in \{0, 1\}^2$ we have

$$\begin{aligned}
\mathbb{E}[W_0(i_0) \times W_1(i_1)|X_0] &= P(W_j(i_j) = 1 \forall j = 0, 1|X_0) \\
&= P(W_0(i_0) = 1|X_0) \times P(W_1(i_1) = 1|X_0) \\
&= \mathbb{E}[W_0(i_0)|X_0] \times \mathbb{E}[W_1(i_1)|X_0].
\end{aligned}$$

On the other hand by Assumption 4,

$$\begin{aligned}
P(W_j(1) > W_j(0)|X_0) &= 1 - P(W_j(0) = 1|X_0) - P(W_j(1) = 0|X_0) \\
&= P(W_j(1) = 1|X_0) - P(W_j(0) = 1|X_0) \\
&= \mathbb{E}[W_j(1)|X_0] - \mathbb{E}[W_j(0)|X_0].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P(W_j(1) > W_j(0) \forall j = 0, 1|X_0) \\
&= \left(\mathbb{E}[W_0(1)|X_0] - \mathbb{E}[W_0(0)|X_0] \right) \times \left(\mathbb{E}[W_1(1)|X_0] - \mathbb{E}[W_1(0)|X_0] \right) \\
&= \mathbb{E}[W_0(1)W_1(1) - W_0(0)W_1(1) - W_0(1)W_1(0) + W_0(0)W_1(0)|X_0].
\end{aligned}$$

Thus, we can determine the probability $P(W_j(1) = 1, W_j(0) = 0 \forall j|X_0)$ if each term $\mathbb{E}[W_0(i_0)W_1(i_1)|X_0]$ is determined.

To this end, a direct application of Theorem 3.1 in Pham (2016) implies

$$\mathbb{E}[W_0(1)W_1(1)|X_0] = \mathbb{E}\left[W_0W_1 \times \frac{Z_0Z_1}{P(Z_0 = 1|X_0)P(Z_1 = 1|X_0, Z_0, X_1, Y_1)} \middle| X_0\right].$$

Similarly, $P(W_j(1) > W_j(0) \forall j = 0, 1|X_0)$ is identified.

Appendix B Derivation of Equation (3.4)

$$\begin{aligned}
& \mathbb{E} \left[u_{10}(\cdot) - u_{01}(\cdot) | X_0, W_j(1) > W_j(0) \forall j = 0, 1 \right] \\
= & \mathbb{E} \left[u(\cdot) | X_0, W_0 = 1, W_1 = 0, W_j(1) > W_j(0) \forall j = 0, 1 \right] \\
& - \mathbb{E} \left[u(\cdot) | X_0, W_0 = 0, W_1 = 1, W_j(1) > W_j(0) \forall j = 0, 1 \right] \\
= & \mathbb{E} \left[u(\cdot) | X_0, Z_0 = 1, Z_1 = 0, W_j(1) > W_j(0) \forall j = 0, 1 \right] \\
& - \mathbb{E} \left[u(\cdot) | X_0, Z_0 = 0, Z_1 = 1, W_j(1) > W_j(0) \forall j = 0, 1 \right] \\
= & \frac{1}{P(W_j(1) > W_j(0) \forall j = 0, 1 | X_0)} \left[\mathbb{E} \left[\kappa u(\cdot) \frac{W_0(1 - W_1)}{P(Z_0 = 1, Z_1 = 0 | X_0, X_1)} | X_0 \right] \right. \\
& \left. - \mathbb{E} \left[\kappa u(\cdot) \frac{(1 - W_0)W_1}{P(Z_0 = 0, Z_1 = 1 | X_0, X_1)} | X_0 \right] \right]. \quad (10)
\end{aligned}$$

Appendix C Derivation of Equation (4.1)

First DeMorgan's Law gives

$$\begin{aligned}
& \mathbb{E} \left[u(\cdot) | X_0, W_j(1) > W_j(0), \forall j \leq \tau, W_{\tau+1}(1) = W_{\tau+1}(0) = i \right] \\
& \times \left(\prod_{j=0}^{\tau} P(W_j(1) > W_j(0) | X_0) \right) - \mathbb{E} \left[u(\cdot) | X_0, W_{\tau+1}(1) = W_{\tau+1}(0) = i \right] \\
= & \sum_{k=1}^{\tau} (-1)^k \sum_{\substack{i_1, \dots, i_k \\ \in \{0,1\}}} \sum_{\substack{j_1 < \dots < j_k \\ \in \{0,1, \dots, \tau\}}} \mathbb{E} \left[u(\cdot) | X_0, W_{j_m}(1) = W_{j_m}(0) = i_m \forall m \in \{1, \dots, k\}, \right. \\
& \left. W_{\tau+1}(1) = W_{\tau+1}(0) = i \right].
\end{aligned}$$

Thus the expected utility of a full-complier is

$$\begin{aligned}
& \mathbb{E}\left[u(\cdot)|X_0, W_j(1) > W_j(0) \forall j\right] \\
&= \frac{1}{P(W_j(1) > W_j(0) \forall j|X_0)} \times \left(\mathbb{E}\left[u(\cdot)|X_0\right] + \right. \\
&\quad \left. \sum_{\tau=1}^T (-1)^\tau \sum_{\substack{i_1, \dots, i_\tau \\ \in \{0,1\}}} \sum_{\substack{j_1 < \dots < j_\tau \\ \in \{0,1, \dots, T\}}} E_{X_1, \dots, X_{j_\tau}|X_0} \left[\frac{\mathbb{E}\left[u(\cdot) \prod_{t=1}^{\tau} K_{j_t, i_t} | X_0, \dots, X_{j_\tau}\right]}{P(Z_{j_1} = i_1, \dots, Z_{j_\tau} = i_\tau | X_0, \dots, X_{j_\tau})} \right] \right) \\
&= \frac{\mathbb{E}[\kappa u(\cdot)|X_0]}{P(W_j(1) > W_j(0) \forall j|X_0)},
\end{aligned}$$

Appendix D Derivation of Equation (4.3)

By Assumption 2.5, we have

$$\begin{aligned}
& P(W_j(1) > W_j(0) \forall j | X_0) \\
&= P(W_0(1) > W_0(0) | X_0) \cdot P(W_j(1) > W_j(0) \forall j \geq 1 | W_0(1) > W_0(0), X_0) \\
&= P(W_0(1) > W_0(0) | X_0) \cdot P(W_j(1) > W_j(0) \forall j \geq 1 | X_0) \\
&= P(W_0(1) > W_0(0) | X_0) \cdot P(W_1(1) > W_1(0) | X_0) \\
&\quad \times \mathbb{E}_{\overline{O}_1, Z_0 | X_0} [P(W_j(1) > W_j(0) \forall j \geq 2 | W_1(1) > W_1(0), \overline{O}_1, Z_0)] \\
&= P(W_0(1) > W_0(0) | X_0) \cdot P(W_1(1) > W_1(0) | X_0) \\
&\quad \times \mathbb{E}_{\overline{O}_1, Z_0 | X_0} [P(W_j(1) > W_j(0) \forall j \geq 2 | \overline{O}_1, Z_0)] \\
&= P(W_0(1) > W_0(0) | X_0) \cdot P(W_1(1) > W_1(0) | X_0) \cdot P(W_j(1) > W_j(0) \forall j \geq 2 | X_0).
\end{aligned}$$

Continuing this process, we obtain the result.

Appendix E Proof of Theorem (4.3)

First note that under Assumption 5, we have can expand $P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0)$ as a product form

$$\prod_{j \in \mathcal{T}_c} P(W_j(1) > W_j(0)|X_0) \prod_{j \in \mathcal{T}_n^0} P(W_j(1) = W_j(0) = 0|X_0) \prod_{j \in \mathcal{T}_n^1} P(W_j(1) = W_j(0) = 1|X_0).$$

Next by Assumption 4,

$$\begin{aligned} P(W_j(1) = W_j(0) = 0|X_0) &= P(W_j(1) = 0|X_0) = \mathbb{E}[1 - W_j(1)|X_0]; \\ P(W_j(1) = W_j(0) = 1|X_0) &= P(W_j(0) = 1|X_0) = \mathbb{E}[W_j(0)|X_0]. \end{aligned}$$

We can rewrite $P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0)$ as

$$\begin{aligned} & \prod_{j \in \mathcal{T}_c} \left(\mathbb{E}[W_j(1)|X_0] - \mathbb{E}[W_j(0)|X_0] \right) \prod_{j \in \mathcal{T}_n^0} \left(\mathbb{E}[1 - W_j(1)|X_0] \right) \prod_{j \in \mathcal{T}_n^1} \left(\mathbb{E}[W_j(0)|X_0] \right) \\ &= \left(\sum_{(i_j)_{j \in \mathcal{T}_c} \in \{0,1\}^{|\mathcal{T}_c|}} (-1)^{|\mathcal{T}_c| - \sum_{j \in \mathcal{T}_c} i_j} \cdot \mathbb{E} \left[\prod_{j \in \mathcal{T}_c} W_j(i_j) \middle| X_0 \right] \right) \\ & \quad \cdot \mathbb{E} \left[\prod_{j \in \mathcal{T}_n^0} (1 - W_j(1)) \middle| X_0 \right] \mathbb{E} \left[\prod_{j \in \mathcal{T}_n^1} (W_j(0)) \middle| X_0 \right]. \end{aligned}$$

Applying Lemma 1, we obtain the identification result for general compliance type $P(\mathcal{T}_c, \mathcal{T}_n^0, \mathcal{T}_n^1|X_0)$.

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